THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2016–2017)
Introduction to Topology
Exercise 3 Base of Topology

Remarks. Many of these exercises are adopted from the textbooks (Davis or Munkres).

1. What is the topology generated by all closed intervals in $\mathbb{R}$?

2. Which subset(s) of $\mathcal{P}(X)$ will generate the indiscrete topology?

3. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for two topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$ of X respectively. Is $\mathcal{B}_1 \cap \mathcal{B}_2$ a base for some topology?

4. If $\mathcal{B}_1 \subset \mathcal{B}_2$ are two bases for the same topology $\mathfrak{T}$ of X, and if $\mathcal{B}_1 \subset \mathcal{A} \subset \mathcal{B}_2$, can we say that $\mathcal{A}$ is a base for $\mathfrak{T}$?

5. Let $\mathcal{S} \subset \mathcal{P}(X)$ and $\mathfrak{T}_\mathcal{S}$ be the topology generated by $\mathcal{S}$. Show that it is also the smallest topology of X containing $\mathcal{S}$.

6. Let $\mathcal{S}$ and $\mathcal{B}$ be a subbase and base of a topological space $(X, \mathfrak{T})$ and $A \subset X$. Is there a natural way to create a corresponding subbase and base of the induced topology $\mathfrak{T}|_A$?

7. This exercise makes reference to $\mathfrak{T}_1$ on $\mathcal{C}([a,b],\mathbb{R})$ in HW01. For an open set $U \subset \mathbb{R}$ and a closed interval $K \subset [a,b]$, define a set

$$W_{(K,U)} = \{\, f \in \mathcal{C}([a,b],\mathbb{R}) : \ \mathrm{graph}(f|_K) \subset K \times U \subset [a,b] \times \mathbb{R} \,\} .$$

   Let $\mathfrak{T}_2$ be the topology generated by $\big\{\, W_{(K,U)} : \text{ closed interval } K \subset [a,b], \text{ open } U \subset \mathbb{R} \,\big\}$. What is the relation between $\mathfrak{T}_1$ and $\mathfrak{T}_2$?

8. Let X be a totally ordered set (or called linearly ordered or simply ordered). That is, there is a relation $<$ on X such that any $x, y \in X$ must have $x < y$ or $y < x$. In such a set, we may naturally define intervals (a,b), etc.

   Assume that X has neither largest nor smallest elements. Show that $\mathcal{B} = \{\, (a,b) : \ a, b \in X \,\}$ is a base of a topology. What if X has a largest element M or smallest element m?

   Remark. This is called the *order topology*.

9. Let $(X, \mathfrak{T})$ be a topological base that has a countable base $\mathcal{C}$. Show that every base $\mathcal{B}$ has $\mathcal{B}_c \subset \mathcal{B}$ such that $\mathcal{B}_c$ is a countable base.

10. Fill in carefully the details in the proofs of "A separable metric space is of second countable".

11. What are the typical dense subsets in the lower limit topology, cofinite topology, and $\mathfrak{T}_{cf0}$ in HW01?

12. This exercsie shows why countability is important. Let $(X, \mathfrak{T})$ have a countable base $\mathcal{B}$. Then every uncountable set $A \subset X$ has uncountably many cluster points.

13. A topological space is called Lindelof if every open cover has a countable subcover. Show that a Lindelof metric space is of second countable.

    Remark. This countability condition of Lindelof is somehow more related to compactness. An *open cover* of X is a subset of the topology, $\mathcal{C} \subset \mathfrak{T}$, such that $\cup\mathcal{C} = X$. A subset $\mathcal{E} \subset \mathcal{C}$ is called a *subcover* if it is also an open cover.

14. Let $(X, \mathfrak{T})$ be a topological space and $A \subset X$ be given the topology (so-called induced or relative) $\mathfrak{T}|_A$ where

    $$\mathfrak{T}|_A = \{ A \cap U \ : \ U \in \mathfrak{T} \} .$$

    (a) If X is second-countable or first-countable, then so is A.

    (b) If X is Lindeloff and A is closed, then A is also Lindeloff.

    (c) What about A if X is separable?